

# Math 206A Lecture 23 Notes

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## 1 Steinitz's Theorem

### 1.1 Steinitz's theorem

**Theorem 1.1** (Steinitz, 1928). *For every 3-connected, planar graph  $G = (V, E)$ , there exists a convex polytope  $P \subseteq \mathbb{R}^3$  such that  $G$  is a graph of  $P$ .*

Last time, we talked about when  $G$  is a plane triangulation. We also discussed the following open problem:

**Theorem 1.2** (quantitative Steinitz problem). *Let  $M(n)$  be the minimum  $m$  such that for all  $G = (V, E)$  with  $|V| = n$ , there exists a polytope with nonnegative integer coordinates  $\leq m$  with graph  $G$ . Then  $M(n) < n^c$  for some constant  $c$ .*

It is currently known that  $M(n) \leq 147.7^n$ .

### 1.2 Ideas in the development of Steinitz's theorem

The idea is due to Maxwell (the physicist) in the 1860s. Suppose we have a polytope shaped like a dome, and we project it down to get a graph.

**Lemma 1.1.** *There exists a weight function  $w : E \rightarrow \mathbb{R}_+$  such that for all  $v \in V \setminus \partial V$ ,  $\sum_{(v,v') \in E} w(v,v')vv' = 0$ , where  $vv'$  is the vector from  $v$  to  $v'$ .*

If you think of each edge as a spring, this says that this is an equilibrium state of the springs. Suppose we have a vertex  $v$ ; what does the equilibrium say? If we take the edge vectors connected to  $v$ , and take a perpendicular vector to each edge vector (oriented in circular fashion), we get a polygon enclosing the vertex  $v$ . How do we get these perpendicular vectors? This is left to the reader.<sup>1</sup>

**Lemma 1.2** (Cremona). *Maxwell's map from polytopes to networks in equilibrium is invertible.*

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<sup>1</sup>When Maxwell gave his lecture, he just read the paper to the audience without any pictures. Everything was left to the imagination. These notes are like that, as well.

Here is the next idea in the proof.

**Theorem 1.3** (Tutte). *For every 3-connected, planar graph  $G$ , there exists an embedding/drawing of  $G$  in  $\mathbb{R}^2$  such that all faces are convex polygons.*

*Proof.* Take  $w = 1$  (but this works for all  $w > 0$ ). There exists an equilibrium (spring) embedding of  $G$ . Pin down the boundary vertices. From a physics perspective, if we let the spring network go, the springs will move around until they reach an equilibrium. If we define the energy function  $\mathcal{E} = \sum_{e \in E} w(e)|e|^2$ , then Tutte is saying that there exists a (unique) minimum of  $\mathcal{E}$ .  $\square$

There is a hole the size of the Pacific ocean in the above proof. What is the issue? We need an equilibrium embedding, not just an equilibrium. We need to make sure the graph is planar. So we need to use Kuratowski's theorem, that every planar graph contains  $K_5$  or  $K_{3,3}$ . We also need to use the 3-connectedness of the graph. The actual argument is very complicated.

Here is the last step in the proof of Steinitz's theorem. The algorithm is that we start with a graph, get the Tutte embedding and use Cremona's lemma to get a polytope.

**Lemma 1.3.** *The Tutte spring embedding can be realized in the box  $[1, M]^2$ , where  $M$  is proportional to the number of spanning trees in  $G$ .*

*Proof.* The expression  $\sum_{(v,v') \in E} w(v,v')vv' = 0$  is the determinant of a matrix.  $\square$

**Lemma 1.4.** *The number of spanning trees in a planar graph  $G = (V, E)$  is  $\leq 5.3^n$*

This gives an idea of how to get bounds in the quantitative problem.